Similarity Solutions for Non-Newtonian Fluids

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Similarity transformations are possible for power law fluids in the following systems: momentum transfer in general Falkner-Skan flows and Goldstein flows; and momentum and energy transfer in forced convection about a right angle wedge, natural convection with constant heat flux at the boundary surface, and general Falkner-Skan flows with nonconstant heat conductivity and a restricted boundary temperature distribution. Similarity transformations are also possible for momentum and heat transfer of Eyring viscous fluids about a right angle wedge. A numerical solution is obtained for forced convection of power law fluids about a right angle wedge with an isothermal surface. From the numerical results an approximate expression is obtained for the local Nusselt number.

The classical theory of fluid mechanics is based upon the hypothesis of a linear relationship between stress and strain rate:

$$\tau = -\mu \frac{\partial U}{\partial y} \tag{1}$$

Fluids with properties different from that described by Equation (1), called non-Newtonian fluids, are generally subdivided into two categories: viscoelastic fluids, where the stress depends upon both the instantaneous strain rate and the past strain history; and viscoinelastic fluids, where the stress depends upon the current state of strain only. The classical Newtonian fluids are included here as a special case.

In this paper we concentrate our discussion upon the non-Newtonian viscoinelastic fluids. Three kinds of empirical equations are usually used in approximating experimental data for such fluids. These are

1. Power law fluids:

$$\tau = -K\left(\frac{\partial U}{\partial y}\right)^n \tag{2}$$

2. Eyring viscous fluids:

$$\tau = -K_1\left[1 - \frac{\sinh^{-1}\left(K_2\dfrac{\partial U}{\partial y}\right)}{K_3\left(\dfrac{\partial U}{\partial y}\right)}\right] \tag{3}$$

3. Bingham plastics (which do not concern us here) where K, n, K_1, K_2, and K_3, are empirical constants. These expressions can be used only for rectilinear flows. Extension to multidimensional and curvilinear flows is not difficult. From Bird, Stewart, and Lightfoot (*1*) we find that a useful generalization of Equations (2) and (3) may be written as

$$\tau^{ij} = p\delta^{ij} - K\,(II)^{(n-1)/2}B^{ij} \tag{4}$$

$$\tau^{ij} = p\delta^{ij} - K_1\left\{1 - \frac{\sinh^{-1}[K_2(II)^{1/2}]}{K_3(II)^{1/2}}\right\}B^{ij} \tag{5}$$

for power law and Eyring viscous fluids, respectively. Thus the effective viscosity $\overline{\mu}$ becomes

$$\overline{\mu} = K\,(II)^{(n-1)/2} \tag{6}$$

$$\overline{\mu} = K_1\left\{1 - \frac{\sinh^{-1}[K_2(II)^{1/2}]}{K_3(II)^{1/2}}\right\} \tag{7}$$

for the two models. Here $II = \frac{1}{2}B_j^i B_i^j$ is the second invariant of the strain rate tensor B^{ij} (*1*).

By applying the constitutive Equations (4) and (5), the governing equations for the transportation of mass, momentum, and energy can be formulated and solved for given geometries. Most research has dealt with interior flow in pipe lines, annuli, or channels; exterior flows past immersed bodies have been discussed in only a few cases. For Newtonian fluids two methods are usually used to solve exterior flows: the similarity method and the integral method. The similarity method involves the determination of similarity variables which reduce the system to ordinary differential equations. Probably the first analysis of this type was given by Acrivos, Shah, and Peterson (*2*). In this paper forced convection is considered and similarity variables are given for general wedge flow of power law fluids. The transformation is applicable only to the momentum equation. For the determination of heat transfer, an approximate method was suggested for very large Prandtl numbers, where the thermal boundary layer is much thinner than the velocity boundary layer. As a continuation of this paper, Acrivos (*3*) made another analysis for laminar natural convection. Again a large Prandtl number was assumed so that the momentum transport terms in the equation of motion could be neglected. By realizing that the boundary-layer assumption is an asymptotic approximation, further purposeful neglect of certain terms seems to be permitted. However, from a mathematical viewpoint, the possibility of solving these equations without such assumptions is a desirable goal.

Two-dimensional jet flow has been discussed by Gutfinger and Shinnar (*4*) and independently by Kapur (*5*). This research deals only with the momentum transfer and closed form solutions have been given. Schowalter (*6*) illustrated the possibility of reducing the three-dimensional boundary-layer equations to ordinary differential equations subject to the limitations of flat plate flow. All this research considered power law fluids only. More recently, Metzner and White (*7*) contributed to the similarity transformation for the momentum equation of viscoelastic flow. They conclude that the possibility of solving these systems is limited.

The integral method, originated in Pohlhausen's application to Newtonian fluid systems, was also discussed by Acrivos, Shah, and Peterson (*2*). Their comparison with the analytic solution showed poor accuracy for predicting the drag coefficient, and hence is not applicable to energy transfer solutions which depend strongly upon the solutions of momentum transfer.

The major difficulty in analyzing non-Newtonian fluid systems results from the nonlinearities in the equations of motion. This limits the applicability of similarity variables to the energy equation. Classical methods for finding similarity variables fail to indicate the possible cases.

The classical method of separation of variables for determining similarity variables requires exhaustive effort, and owing to its complexity does not appear to provide a general procedure for complex problems.

Birkhoff (*8*), in initiating the applications of group theory to fluid mechanics, paved the way for general similarity procedures. Building on this work, Michal (*9*) and Morgan (*10*) completed the structure of the theory of reduction in number of independent variables. More recently, Ames (*11*) applied the method for reducing more than one variable simultaneously by composing a multiparameter group from one-parameter groups. In his paper (*12*) the convenience and simplicity of this method is noted. The one-parameter group method is also discussed by Hansen (*13*). This method transfers the problem of searching for similarity variables to that of solving for the invariant conditions of a system of differential equations under a certain group of continuous transformations. The similarity variables are determined from the absolute invariants of the subgroup consisting of transformations of independent variables. The new functions are, accordingly, obtained from the remaining elements of the group.

FORMULATION OF FUNDAMENTAL EQUATIONS

The following assumptions are made in this paper.

1. The fluid is assumed incompressible and viscoinelastic.
2. The flow is two-dimensional and steady.
3. The constant pressure specific heat C_p is assumed to be constant with respect to temperature or velocity changes.
4. The generally assumed constant heat conductivity is first used. (However, as will be shown later, in many cases, the nonconstant conductivity, especially as a power function of temperature, is allowable for the derivation of similarity variables with various boundary conditions.)
5. The power law description of viscosity is used. Cases of Eyring viscous fluids will be briefly discussed for their possible similarity transformation forms.

For a fluid passing by a submerged body, the viscous effect is significant only in a boundary layer adjacent to the body surface. In this case, the well-known flow equations are simplified to describe the fluid behavior in the boundary layer. A magnitude analysis similar to that used in Newtonian fluid flows (*14*) has been applied to the power law model fluids by Shah (*14*) and by Schowalter (*6*) for two- and three-dimensional cases. The resulting equations for two-dimensional flow are:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{8}$$

$$\rho u^* \frac{\partial u^*}{\partial x^*} + \rho v^* \frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + K \frac{\partial}{\partial y^*}\left(\left| \frac{\partial u^*}{\partial y^*} \right|^{n-1} \frac{\partial u^*}{\partial y^*} \right) \tag{9}$$

and

$$\frac{\partial p^*}{\partial y^*} = 0 \tag{10}$$

For the flow outside the boundary layer, the velocity gradient $(\partial u^*/\partial y^*)$ is assumed small so that Equation (9) reduces to

$$U_e^* \frac{dU_e^*}{dx^*} = -\frac{1}{\rho}\frac{dp^*}{dx^*} \tag{11}$$

where U_e^* is the x^* component of the main stream velocity.

Falkner and Skan (*14*) considered cases of wedge flows for which U_e^* can be expressed as

$$U_e^* = U_i (x^*)^m \tag{12}$$

where U_i is a constant characterizing the flow magnitude and m is related to the wedge angle β by

$$m = \frac{\beta}{2-\beta} \tag{13}$$

Goldstein (*16*) suggested another type of main stream flow:

$$U_e^* = U_i e^{Cx^*} \tag{14}$$

Both of these cases will be considered herein. For generality we shall write

$$U_e^* = U_e^* \ (x^*) \tag{15}$$

in the following derivations.

Upon subjecting the energy equation to a similar asymptotic analysis, the boundary-layer form becomes

$$\rho C_p u^* \frac{\partial T^*}{\partial x^*} + \rho C_p v^* \frac{\partial T^*}{\partial y^*} = \frac{\partial}{\partial y^*}\left(k \frac{\partial T^*}{\partial y^*} \right) + \dot{Q} + \bar{\mu}\left(\frac{\partial u^*}{\partial y^*} \right)^2 \tag{16}$$

The conductivity k is usually assumed constant.

DIMENSIONLESS EQUATIONS

For low-velocity forced convection the basic equations take the form of Equations (8), (9), and (11) and

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{k}{\rho C_p}\frac{\partial^2 T^*}{\partial y^{*2}} \tag{17}$$

These are reduced to dimensionless form by introducing the new variables:

$$x = x^*/L,\ u = u^*/U_i,\ U_e = U_e^*/U_i,$$
$$\theta = (T^* - T_w^*)/(T_e^* - T_w^*)$$
$$N_{Re} = \rho U_i^{2-n} L^n/K,\ N_{Pr} = C_p U_i \rho L/k \ N_{Re}^{2/(1+n)}$$
$$v = N_{Re}\ v^*/U_i,\ y = N_{Re}^{1/(n+1)}\ y^*/L$$

whereupon our basic equations become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{18}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_e \frac{dU_e}{dx} + \frac{\partial}{\partial y}\left(\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right) \tag{19}$$

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \frac{1}{N_{Pr}}\frac{\partial^2 \theta}{\partial y^2} \tag{20}$$

The natural convection equations can be treated in a similar manner. This was done by Acrivos (*3*) and the resulting equations are Equations (18), (20), and

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_e \frac{dU_e}{dx} + \frac{\partial}{\partial y}\left(\left| \frac{\partial u}{\partial y} \right|^{n-1} \frac{\partial u}{\partial y} \right) + \theta \tag{21}$$

THE TRANSFORMATION GROUP METHOD

A summary of the transformation group method, as described by Ames (*11*), is given below for completeness.

A system of N partial differential equations

$$\Phi_j\ (x_i, y_j) = 0, \quad j = 1, \ldots, N, \tag{22}$$

where x_i, $i = 1, \ldots, M$; y_j, $j = 1, \ldots, N$ are independent and dependent variables, respectively, is said to be conformally invariant under a one-parameter group G defined as

$$G\begin{cases}\overline{x_i} = \overline{x_i}\,(x_1, \ldots, x_N;\ a)\\ \overline{y_j} = \overline{y_j}\,(y_j\,;\,a)\end{cases} \tag{23}$$

where a is a real positive number, if

$$\Phi_j\,(x_i\,;\,y_j) = \phi_j\,(\overline{x_i}\,;\,a)\ \overline{\Phi_j}\,(\overline{x_i}\,;\,\overline{y_j}) \tag{24}$$

Equations (22) are said to be constant conformally invariant under G if ϕ_j is a function of a only, and absolute constant conformally invariant if $\phi_j = 1$.

Theorem

If Equations (22) are constant conformally invariant under G, then the invariant solutions of (22) are the solutions of the following system:

$\Psi_j\,(\eta_1, \ldots, \eta_{M-1};\ f_1, \ldots, f_N) = 0,\ j = 1, \ldots, N$

where $\eta_1 \ldots, \eta_{M-1}$ and f_j are the $M + N - 1$ functionally independent absolute invariants of G which can be determined by the following principles:

1. If G_1 is defined as

$$G_1\begin{cases}\overline{x_i} = a^{\alpha_i}x_i,\ i = 1, 2, \ldots, M\\ \overline{y_j} = a^{\lambda_j}y_j,\ j = 1, 2, \ldots, N\end{cases} \tag{25}$$

then

$$\eta_i = \frac{x_i}{{x_k}^{\alpha_i/\alpha_k}},\quad i \neq k$$

$$f_j = \frac{y_j}{{x_k}^{\lambda_j/\alpha_k}} \tag{26}$$

where $\alpha_k \neq 0$ and α_i, λ_j are to be determined under conformally invariant conditions of Equations (22).

2. If in G_1, $\alpha_k = 0$, then the group

$$G_2\begin{cases}\overline{x_k} = x_k + \ln a\\ \overline{x_i} = a^{\alpha_i}x_i,\ i \neq k\\ \overline{y_j} = a^{\lambda_j}y_j\end{cases} \tag{27}$$

yields

$$\eta_i = \frac{x_i}{\exp\,(\alpha_i x_k)}$$

$$f_j = \frac{y_j}{\exp\,(\lambda_j x_k)} \tag{28}$$

For G_2, an alternative form is given by Hansen (*13*) as

$$G_2\begin{cases}\overline{x_k} = x_k + \beta_k a\\ \overline{x_i} = e^{\beta_i a}\,x_i,\ i \neq k\\ \overline{y_j} = e^{\delta_j a}\,y_j\end{cases} \tag{29}$$

The applications of one-parameter group theory for simultaneous elimination of more than one variable and extensions to two or more parameter groups are given by Ames (*11, 12*).

With the above methods, most physical systems can be analyzed for possible similarity solutions. The group methods will be applied herein to power law, non-Newtonian, boundary-layer flow systems. Detailed procedures will be carried out for the first two cases. Extensions to more complicated cases, which make use of earlier results, will be treated in a similar manner. Further extension to non-Newtonian flow of other models will also be discussed.

The arguments are restricted to Cartesian two-dimensional steady flows. Similarity variables, that is, invariants of corresponding groups, are derived; the ordinary differential equations obtained from the transformations are tabulated in Table 1.

TABLE 1. IMPORTANT SIMILAR EQUATIONS

Flow System	Similar Variables	Group Used	Similar Differential Equations	Boundary Conditions	Remarks
Flat Plate Mom. Transfer	(37)	G_1	$n(n+1)f''' + (f'')^{2-n} f = 0$		Solved by Acrivos and co-workers [2]
Two Dimens. Jet Flow	(41)	G_1	$f'^2 + ff'' = \frac{d}{d\eta}(\|f''\|^{n-1}f'')$		Solved by Kapur [5] and by Gutfinger and Shinnar [4]
Falkner-Skan Flow	(45)	G_1	$f'^2 - \frac{(2n-1)m+1}{m(n+1)} ff'' = 1 + \frac{1}{m}\frac{d}{d\eta}(f''^n)$	$Ue = x^m$	
Goldstein Flows	(55)	G_2	$f'^2 - \frac{1-2n}{n+1} ff'' = 1 + \frac{1}{C}\frac{d}{d\eta}(f''^n)$	$Ue = e^{Cx}$, $C \neq 0$	
Wedge Flow Forced Convection	(57)	G_1	$f'^2 - 2ff'' = 1 + 3\frac{d}{d\eta}(f''^n)$ $g'' + N_{Pr}(\frac{2}{3}fg' - t f'g) = 0$	$Ue = x^{\frac{1}{3}}$ $\theta_B = x^t$	$t = 0$, Isothermal, Solved in this paper $t = \frac{1}{3}$, Const. heat flux $t = \frac{2}{3}$, Heat generating, high velocity flows with terms added
Natural Convection	(57) (61)	G_1	$f'^2 - 2ff'' = 3\frac{d}{d\eta}(\|f''\|^{n-1}f'') + 3g$ $g'' + \frac{1}{3}N_{Pr}(2fg' - f'g) = 0$	$\theta_B = x^{\frac{1}{3}}$	The heat flux at wall is limited to be constant
Flat Plate Forced Conv. non-constant Conductivity	(72) $m = 0$	G_1	$n(n+1)f''' + (f'')^{2-n} f = 0$ $\frac{1-n}{(r-1)(n+1)} f'g - \frac{1}{n+1} fg' = \frac{1}{N_{Pr}}\frac{d}{d\eta}(g^{r-1}g')$	$\theta_B = x^{\frac{n-1}{(r-1)(n+1)}}$	Falkner-Skan flow can be derived with (72) and $m \neq 0$

STEADY FLOW MOMENTUM TRANSFER

Linear Group

Here, temperature differences are assumed sufficiently small so that their effect on physical properties can be neglected. This system is treated first with the linear group G_1 for simplicity; the equations are (18) and (19). Noting that the velocity gradient in the y direction is always positive, we can, with care, ignore the absolute value sign. A stream function ψ, defined by the relations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{30}$$

is now introduced. In ψ, the momentum equation becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U_e \frac{dU_e}{dx} + \frac{\partial}{\partial y}\left[\left(\frac{\partial^2 \psi}{\partial y^2}\right)^n\right] \tag{31}$$

The one-parameter linear group

$$G_1 \begin{cases} \bar{x} = a^{\alpha_1} x \\ \bar{y} = a^{\alpha_2} y \\ \bar{\psi} = a^{\alpha_3} \psi \\ \bar{U}_e = a^{\alpha_4} U_e \end{cases} \tag{32}$$

is now applied to Equation (31) and the constant conformally invariant condition requirements for (31) lead to the algebraic equations

$$\alpha_1 + 2\alpha_2 - 2\alpha_3 = \alpha_1 + 2\alpha_4 = (2n + 1)\,\alpha_2 - n\alpha_3 \tag{33}$$

This set of equalities consists only of two equations:

$$\alpha_1 + (1 - 2n)\,\alpha_2 + (n - 2)\,\alpha_3 = 0 \tag{34}$$

and

$$\alpha_2 - \alpha_3 + \alpha_4 = 0 \tag{35}$$

We usually solve for all the α_i's in terms of α_j, where α_j is the parameter for the independent variable to be eliminated. It is seen that (34) and (35), containing three unknowns and only two equations, have no unique solution. This situation provides a degree of freedom in fitting boundary conditions or other restrictions.

For example, the flat plate flow has U_e as a constant, hence $(dU_e/dx) = 0$. In this case $\alpha_4 = 0$. From Equations (34) and (35) we see that

$$\alpha_2 = \alpha_3 = \alpha_1/(n + 1) \tag{36}$$

This relation leads to the invariants of G_1 as

$$\eta = yx^{-1/(n+1)}, \qquad f = \psi x^{-1/(n+1)} \tag{37}$$

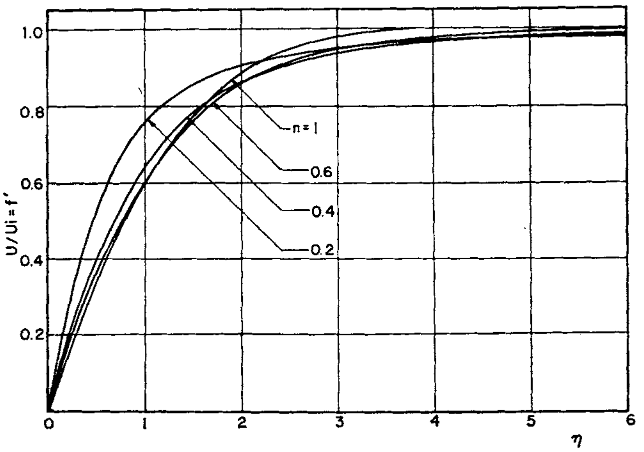

Fig. 1. Velocity profiles for $n \leq 1$.

a form used by Acrivos and co-workers (2) but obtained herein by this alternate method.

Two-dimensional jet flow requires conservation of momentum, that is

$$\int_{-\infty}^{\infty} U^2 dy = \int_{-\infty}^{\infty} \left(\frac{\partial \psi}{\partial y}\right)^2 dy = \text{const.} \tag{38}$$

Upon application of G_1 we have

$$\alpha_2 - 2\alpha_3 = 0 \tag{39}$$

and by solving (34) and (39) we obtain

$$2\alpha_3 = \alpha_2 = \frac{2}{3n}\alpha_1 \tag{40}$$

Hence the resulting invariants are

$$\eta = yx^{-1/3n}, \qquad f = \psi x^{-2/3n} \tag{41}$$

a result previously given by Gutfinger and Shinnar (*4*) and by Kapur (*5*).

If the general form of Falkner-Skan wedge flow is considered, that is

$$U_e = x^m \tag{42}$$

then it can readily be seen that

$$\alpha_4 = m\alpha_1 \tag{43}$$

By using this relation, the solutions of Equations (34) and (35) are

$$\alpha_2 = [(n - 2)\,m + 1]\,\alpha_1/(n + 1), \qquad \alpha_3 = [(2n - 1)\,m + 1]\,\alpha_1/(n + 1) \tag{44}$$

and the corresponding invariants are

$$\eta = yx^{-\alpha_2/\alpha_1}, \qquad f = \psi x^{-\alpha_3/\alpha_1} \tag{45}$$

Thus it is clearly possible to transform the power law momentum equation to an ordinary differential equation for wedge flows with an arbitrary wedge angle.

An interesting case is that where $m = 1/3$, the transformations are independent of n. This does not appear to have any physical significance, since it is applicable only to restricted geometry, that is, a right angle wedge. However, as will be shown later this special case enables the similarity transformations to apply to the energy equation and also to other systems.

The Spiral Group

As can be seen from Equations (53) and (54), when $\alpha_1 = 0$ the system can also have nontrivial solutions. This case should yield to the spiral group

$$G_2 \begin{cases} \bar{x} = x + \ln a \\ \bar{y} = a^{\alpha_2} y \\ \bar{\psi} = a^{\alpha_3} \psi \end{cases} \tag{46}$$

The term $U_e(dU_e/dx)$ is, for the moment, ignored but will be discussed later. The invariance of Equation (31) yields the same relation as (34), except α_1 is absent; that is

$$(1 - 2n)\,\alpha_2 + (n - 2)\,\alpha_3 = 0 \tag{47}$$

In this case, the absolute invariants of G_2 are

$$\eta = y \exp[-\alpha_2 x], \qquad f = \psi \exp[(1 - 2n)\,\alpha_2 x/(n - 2)] \tag{48}$$

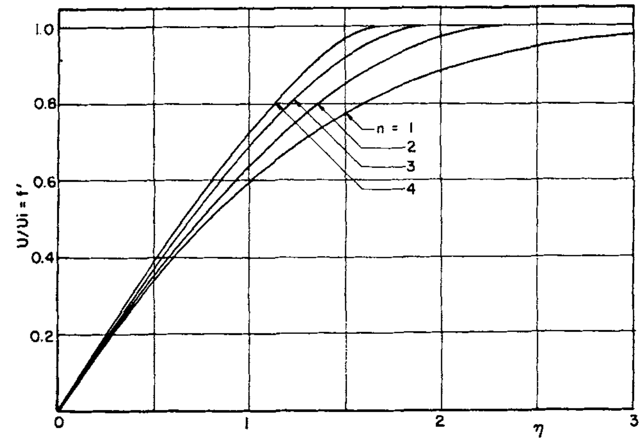

Fig. 2. Velocity profiles for $n \geq 1$.

and these still allow considerable freedom since α_2 is arbitrary.

If the flat plate is considered, the constraint of the boundary condition requires

$$\alpha_2 = \alpha_3 \tag{49}$$

and simultaneous satisfaction of (47) and (49) requires that $n = -1$, or trivially

$$\alpha_2 = \alpha_3 = 0 \tag{50}$$

The power law parameter n is assumed to be between zero and one or greater than one, corresponding to pseudoplastic or dilatant fluids, but negative values have no meaning. The case for which Equation (50) holds yields a constant solution in x for fixed y and corresponds to Couette flow between parallel plates. It will not be considered here.

Returning to the pressure gradient term, it is easily seen that the Falkner-Skan flow is not applicable. Next we consider the Goldstein main stream velocity, that is

$$U_e = \exp(Cx) \tag{51}$$

For this flow the linear group is easily seen to fail; however, with the spiral group we can determine a solution. The pressure gradient term from Equation (51) takes the form

$$U_e \frac{dU_e}{dx} = Ce^{2Cx} \tag{52}$$

By setting Equation (52) into (31) and applying G_2, one obtains

$$2\alpha_2 - 2\alpha_3 = 2C = (2n+1)\,\alpha_2 + n\alpha_3 \tag{53}$$

from which it follows that

$$\alpha_2 = \frac{2-n}{n+1} C, \qquad \alpha_3 = \frac{1-2n}{n+1} C \tag{54}$$

and

$$\eta = \frac{y}{\exp\left\{ \left(\frac{2-n}{n+1}\right) Cx \right\}}, \qquad f = \frac{\psi}{\exp\left\{ \left(\frac{1-2n}{n+1}\right) Cx \right\}} \tag{55}$$

Here C can take any real value except zero. A special case for which the transformations are free of n does not exist. The transformability of the energy equation will be shown to depend upon the value of n.

STEADY FLOW MOMENTUM AND HEAT TRANSFER

By this point the algebraic operations are sufficiently clear that they may be omitted. Only brief summaries will be given.

Forced Convection

When the phenomenon of energy transport is considered, the governing equations are (18), (19), and (20). In the stream function ψ, the system is transformed to Equation (31) and

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{1}{N_{Pr}}\frac{\partial^2 \theta}{\partial y^2} \tag{56}$$

The linear group G_1 [Equation (32)], supplemented with the additional element $\bar{\theta} = a^{\alpha_5}\theta$, is now applied. The resulting absolute invariants are found to be

$$\eta = yx^{-1/3}, \quad f = \psi x^{-2/3}, \quad g = \theta x^{-t} \tag{57}$$

where t is an arbitrary real number. This means that the boundary conditions on θ can be any power function of x, that is

$$\theta_B = x^t \tag{58}$$

including a constant boundary temperature for which $t = 0$ is a special case. However, the main stream flow is restricted to

$$U_e = x^{\alpha_4/\alpha_1} = x^{1/3} \tag{59}$$

which corresponds to right angle flow.

When the spiral group is applied to Equation (56) we find that a similarity solution exists only if $n = 1$, that is, a Newtonian fluid.

Natural Convection

For natural convection, the momentum equation, which is affected by the temperature-induced body force difference, takes the form

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\left| \frac{\partial^2 \psi}{\partial y^2} \right|^{n-1} \frac{\partial^2 \psi}{\partial y^2} \right) + \theta \tag{60}$$

The linear group analysis results in a replacement for g, that is

$$g = \theta x^{-1/3} \tag{61}$$

so that the boundary condition on θ is therefore restricted to

$$\theta_B = x^{1/3} \tag{62}$$

There is no restriction on flow geometry since we usually assume that the fluid is at rest at an infinite distance from the boundary surface.

The spiral group is applicable only to Newtonian fluids with the results as given by Sparrow and Gregg (*17*), with the use of separation of variables.

Miscellaneous Cases of Forced Convection

When a constant heat generation rate $\dot{Q}$ is added to Equation (56), the conformally invariant condition of that equation becomes that of absolute constant conformally invariant; that is

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_5 = 2\,\alpha_2 - \alpha_5 = 0 \tag{63}$$

Since $\alpha_2 = \alpha_1/3$ it follows that

$$\alpha_5 = 2\alpha_1/3 \tag{64}$$

From the energy dissipation term we find the same result. Hence the absolute invariant for θ is

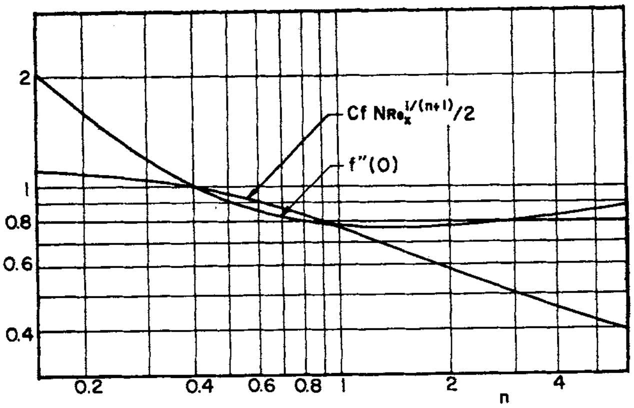

Fig. 3. Relations of drag coefficient and initial slope vs. *n*.

$$g = \theta x^{-2/3} \tag{65}$$

thereby restricting the boundary condition to $\theta_B = x^{2/3}$.

The case of constant heat flux at the boundary surface

$$\left.\frac{\partial \theta}{\partial y}\right|_{y=o} = \text{constant} \tag{66}$$

yields the same result as those for natural convection [Equations (61) and (62)].

EYRING VISCOUS FLOW

In our previous work, the special case for which $2\alpha_2 = \alpha_3 = 2/3\,\alpha_1$ was noted for its independence of nonlinearity and its applicability to the energy equation. This suggests further investigation of the transformabilities of fluids having other empirical models. The Eyring viscosity has the form of Equation (7) and, with the boundary-layer assumption, can be expressed as

$$\bar{\mu} = K_1 \left\{ 1 - \frac{\sinh^{-1}\left[K_2 \dfrac{\partial u}{\partial y}\right]}{K_3 \dfrac{\partial u}{\partial y}} \right\} \tag{67}$$

For simplicity, we will use a more general expression for the effective viscosity, namely

$$\bar{\mu} = \bar{\mu}\left(\frac{\partial u}{\partial y}\right) = \bar{\mu}\left(\frac{\partial^2 \psi}{\partial y^2}\right) \tag{68}$$

If we require $(\partial u/\partial y)$ to be absolute constant conformally invariant, then the term $\bar{\mu}$ is transformable no matter what form it takes. This condition, expressed in algebraic form, again gives the relation $\alpha_3 = 2\alpha_2$ for which both the momentum and energy equations are transformable and the auxiliary conditions should be Equations (58) and (59).

NONCONSTANT HEAT CONDUCTIVITY

All the above analyses have been based upon the assumption of constant physical properties. It has been found that the boundary conditions are unduly restricted. The major cause for these restrictions has been the energy equation. Since the viscosity has been considered nonconstant, it is reasonable to postulate a nonconstant physical property in the energy equation and see if with this the constraints can be relieved. The heat conductivity is a logical choice.

We assume

$$k = k' \,\theta^{r-1} \tag{69}$$

where r is a constant to be determined by the dependence of k upon θ. By inserting this into Equation (56), the energy equation becomes

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{1}{N'_{Pr}}\frac{\partial}{\partial y}\left[\theta^{r-1}\frac{\partial \theta}{\partial y}\right] \tag{70}$$

where N'_{Pr} is defined in terms of k'. The invariant conditions for Equation (70) with G_1 become

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_5 = 2\alpha_2 - r\alpha_5 \tag{71}$$

This and Equations (34) and (35), with $\alpha_4 = m\alpha_1$, yield the solution

$$\begin{aligned} \alpha_2 &= \left[\frac{1+(n-2)m}{n+1}\right]\alpha_1 \\ \alpha_3 &= \left[\frac{1+(2n-1)\,m}{n+1}\right]\alpha_1 \\ \alpha_5 &= \left[\frac{(n-1)-3\,(n-1)m}{(n+1)\,(r-1)}\right]\alpha_1 \end{aligned} \tag{72}$$

Here it is seen, if $r \neq 1$, that m can be any real value including zero; that is, any kind of main stream flow velocity can be applicable provided a uniquely determined boundary condition for θ is prescribed. If $r = 1$, there is no solution for α_5 except when $m = 1/3$ in which case, α_5 is arbitrary.

The restrictions on the velocity boundary conditions are relieved by the assumption of nonconstant heat conductivity. However, restrictions are imposed on the temperature field.

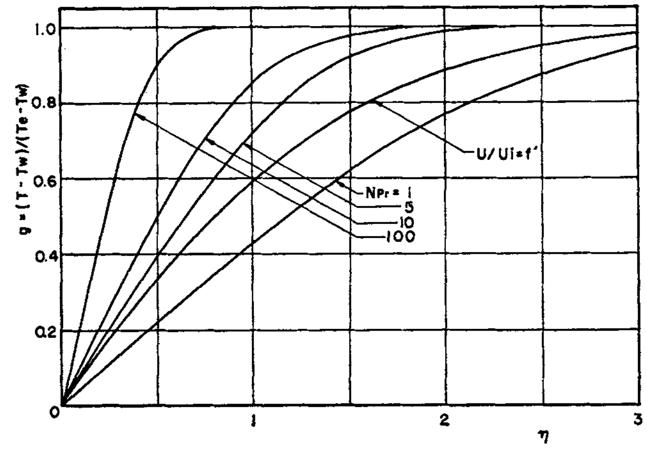

Fig. 4. Comparison of velocity and temperature profiles, *n* = 0.4.

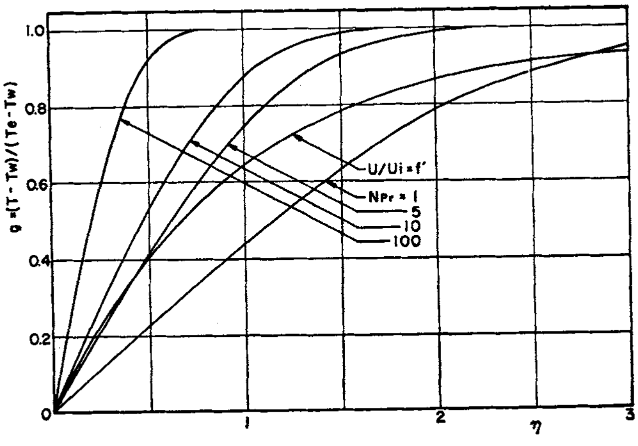

Fig. 5. Comparison of velocity and temperature profiles, *n* = 1.0.

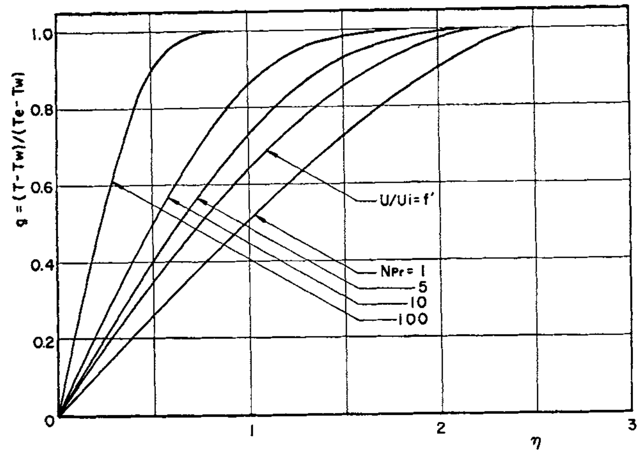

Fig. 6. Comparison of velocity and temperature profiles, $n = 2.0$.

It has been shown that the nonconventional constitutive equation creates substantial restrictions on the system. Similarity transformations are possible only for certain conditional cases. Further analyses on three-dimensional transient states and curvilinear coordinate systems can be carried out, with the group approach.

Table 1 gives the important ordinary differential equations obtained with those similarity variables developed. The forced convection case is chosen for the development of a numerical solution for momentum and heat transfer.

NUMERICAL PROCEDURE

This research was motivated by the desire to obtain a set of suitable similar equations from which we can develop the solution for the momentum and heat transfer in non-Newtonian fluids. Right angle wedge flow is one of the cases where this goal can be achieved. This case also possesses the characteristic of a fairly general temperature boundary condition.

Isothermal wedge flows for power law fluids are chosen for study. The similarity equations take the form

$$(f')^2 - 2ff'' = 1 + 3m(f'')^{m-1}f''' \tag{73}$$

$$g'' + (2/3)\, N_{Pr}\, f\, g' = 0 \tag{74}$$

with boundary conditions

$$f = 0,\ f' = 0,\ g = 0 \text{ at } \eta = 0$$
$$f' = 1,\ g = 1 \text{ as } \eta \to \infty \tag{75}$$

The choice of the boundary condition $g = 0$ at $\eta = 0$ is convenient for comparison of the temperature and velocity profiles.

Equation (73) is not coupled with (74) and hence can be solved independently. A closed form solution for this equation appears to be impossible because of the complex nonlinearity. The method of transforming a boundary value problem to an initial value problem (*18*) is also found to be not applicable to this problem. At this stage numerical procedures are adopted. A continuous analytic continuation scheme (*19*) is employed. An automatic iteration procedure is included in the computer program to determine the correct initial values of f''. Values obtained with this method for Newtonian flow past a flat plate and a wedge have been checked with those tabulated in the literature (*14*), and results show satisfactory agreement. Cases for n ranging from 0.2 to 5 are investigated.

The similar energy equation (74) is seen to be linear and is easily integrated to the form

$$g = \frac{\int_0^\eta \exp\left[-\frac{2}{3} N_{Pr} \int_0^\eta f d\eta\right] d\eta}{\int_0^\infty \exp\left[-\frac{2}{3} N_{Pr} \int_0^\eta f d\eta\right] d\eta} \tag{76}$$

Since only numerical values of f are available, g has to be determined numerically. These calculations are carried out for several cases with Prandtl number ranging from 0.8 to 300.

RESULTS AND DISCUSSION

For the momentum equation, values of f, f', f'', and f''' are obtained for cases with n ranging from 0.2 up to 5. Velocity profiles are plotted in Figures 1 and 2 against the similarity variable η. Figure 1 shows profiles for $n \leqq 1$. It is seen that the initial slope increases and the boundary-layer thickness lengthens as n decreases. In Figure 2 are the cases for $n \geqq 1$. For these the initial slopes increase but boundary-layer thicknesses decrease as n increases.

Since we are interested in the friction resulting from viscosity, a dimensionless drag coefficient has to be deduced from the numerical results mentioned above. The frictional force at the wall takes the form

$$\tau_w = K\left(\frac{\partial u}{\partial y}\right)^n \tag{77}$$

The commonly used drag coefficient is defined as

$$C_f = 2\tau_w^*/\rho U_e^{*2} = 2\, N_{Rex}^{-1/(n+1)} \left(\frac{\partial u}{\partial y}\right)^n \Big|_{y=o}$$
$$= 2N_{Rex}^{-1/(n+1)}\,[f''(0)]^n \tag{78}$$

From this we obtain the relation

$$\frac{1}{2} C_f\ N_{Rex}^{1/(n+1)} = [f''(0)]^n \tag{79}$$

Values of $f''(0)$ and $[f''(0)]^n$ are plotted in Figure 3 vs. n. For $n > 0$, the drag coefficient decreases even though $f''(0)$ increases as n increases.

The numerical solution of Equation (76) generates values of the dimensionless similar temperature and the initial value of temperature slope:

$$g'(0) = \frac{1}{\int_0^\infty \exp\left[-\frac{2}{3} N_{Pr} \int_0^\eta f d\eta\right] d\eta} \tag{80}$$

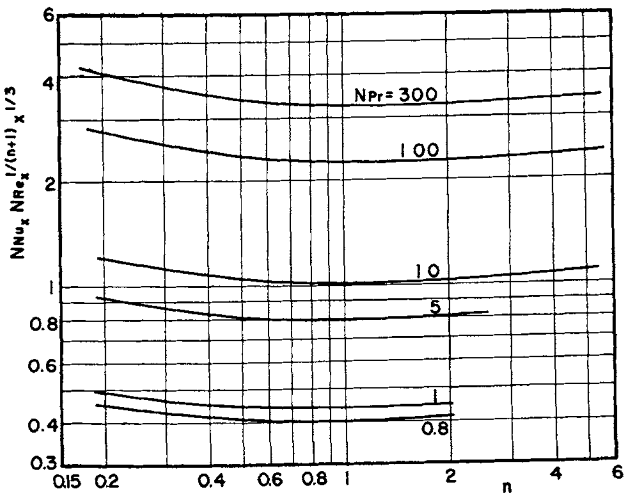

Fig. 7. Relations between N_{Nu} and n.

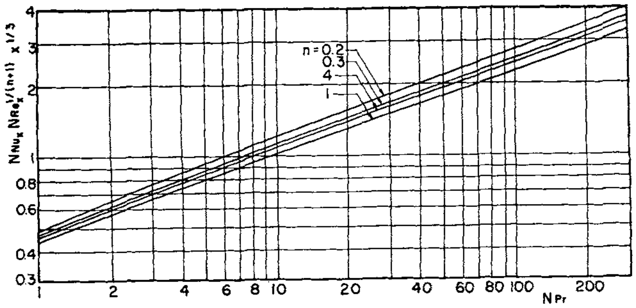

Fig. 8. Relations between N_{Nux} and N_{Pr}.

Figures 4, 5, and 6 show temperature profiles for n equal to 0.4, 1, and 2, respectively. In these figures Prandtl numbers have been chosen as 1, 5, 10, and 100, and corresponding velocity profiles are also plotted. For Prandtl numbers less than 10, the temperature profiles are severely influenced by velocity values. Hence the often used approximation of a straight line velocity profile, within the thermal boundary layer, cannot be accurate even for Prandtl numbers larger than 10. For n greater than 2 and Prandtl numbers less than 5, the thermal boundary layer is much larger than the velocity boundary layer.

The correlation between the Nusselt number and initial temperature slope is derived as follows:

$$N_{Nux} = q(x^*)L/k(T_w^* - T_e^*) = L \frac{\partial T^*}{\partial y^*}\bigg|_{y^*=0} \bigg/ (T_w^* - T_e^*)$$

$$= N_{Rex}^{-1/(n+1)} \frac{\partial \theta}{\partial y}\bigg|_{y=0} = N_{Rex}^{-1/(n+1)}\, x^{-1/3}\, g'(0)$$

Thus

$$x^{1/3}\, N_{Nux}\, N_{Rex}^{1/(n+1)} = g'(0) \tag{81}$$

Actual values of $g'(0)$ are available from the authors. Figure 7 shows the relations between N_{Nux} $N_{Rex}^{1/(n+1)}$ $x^{1/3}$ and the parameter n for different Prandtl numbers. When n departs from one, the influence on heat transfer for n smaller than one is more severe than that for n greater than one. Figure 8 shows the correlation between N_{Nux} $N_{Rex}^{1/(n+1)}$ $x^{1/3}$ and N_{Pr}. These values, plotted on a log-log scale, fall almost on straight lines. Thus we can approximate Nusselt numbers by the equation:

$$N_{Nux} = x^{-1/3}\, N_{Rex}^{-1/(n+1)}\, N^P_{Pr}\, [g'(0)\, |_{N_{Pr}=1}] \tag{82}$$

where P is found to be approximately 1/3. Some values of $g'(0)$, for $N_{Pr} = 1$ are given below:

n	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0
$g'(0)$	0.4886	0.4664	0.4542	0.4474	0.4435	0.4415	0.4407	0.4407	0.4411	0.4747	0.4968

SUMMARY

In this paper the boundary-layer equations for non-Newtonian fluids are investigated. Similarity variables and equations for various flows are obtained by using transformation group methods. Among these, cases that apply to the energy equation are especially to be noted. Transformation forms for Goldstein types of flows, Eyring viscous flows, and nonconstant heat conductivity flows are also given.

Numerical solutions for forced convection flow of power law fluids under right angle wedge geometry have been carried out. These serve not only as useful solutions for this specific geometry, but also as a reference point for approximate solutions of all the numerous unsolvable cases.

NOTATION

a = positive real number
B^{ij} = strain rate tensor
C_p = constant pressure specific heat
C_f = dimensionless drag coefficient
C = real constant characterizing Goldstein flow
d, e, t = arbitrary constants
f = similar function, a similar dimensionless stream function
G = mathematical group
g = similar dimensionless temperature function
g^{ij} = contravariant component of metric tensor
K, K_1, K_2, K_3 = consistency indices of non-Newtonian viscosities
k = heat conductivity
k' = consistency index for nonconstant conductivity
L = characteristic length
m = real constant characterizing the Falkner-Shan flow
M, N = positive integers
N_{Nu} = Nusselt number
N_{Pr} = Prandtl number
N_{Re} = Reynolds number
n = power law exponent
p^* = hydrostatic pressure
p = dimensionless hydrostatic pressure
$\dot{Q}$ = time rate of energy generation per unit volume
q = heat flux
r = constant characterizing the heat conductivity as a function of temperature
U_e = the dimensionless free stream velocity
U^*_e = free stream velocity
U_i = characteristic velocity
$u, \bar{u}$ = dimensionless velocity component in x direction
u^* = velocity component in x^* direction
$v, \bar{v}$ = dimensionless velocity component in y direction
v^* = velocity component in y^* direction
$x, \bar{x}$ = dimensionless space coordinate
x^* = space coordinate
$y, \bar{y}$ = dimensionless space coordinate
y^* = space coordinate

Greek Letters

α, β, γ = parameters
δ^{ij} = Kronecker delta
$\Phi, \bar{\psi}$ = system of functions
$\psi, \bar{\psi}$ = dimensionless stream function
τ^{ij} = stress tensor
μ = Newtonian viscosity
$\bar{\mu}$ = effective viscosity of non-Newtonian fluids
ρ = density
η = similar space variable
Π = second invariant of strain rate tensor
$'$ = differentiation with respect to η

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Entrainment of Air into a Liquid Spray

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An investigation has been carried out into the flow pattern existing in and around a flat spray, particular attention being paid to the region of disintegration of the liquid sheet. The mass of air entrained into the spray, the decay of air velocity along the spray axis, and the spread of the drops in the plane normal to that of the sheet have been related to the operating conditions by equations theoretically derived and experimentally confirmed. Similarities between the characteristics of air entrainment into liquid sprays and into gas jets have been noted.

A spray nozzle is a device which transforms bulk liquid into a spray and which projects the spray in a controlled direction. While many studies have been made of the processes of drop formation, little attention has been given to the subsequent history of the resulting spray and to the way in which it interacts with the surrounding atmosphere.

The aerodynamic properties of sprays are significant in a wide range of applications; for example, in a spray dryer the initial rate of mixing of the spray with the hot gas plays a large part in determining the bulk density of the final dried product. In a combustion chamber the burning characteristics of a fuel spray depends on both the manner in which the air is entrained and mixes with the spray, and on the final mixture ratio. Furthermore, the penetration of the fuel drops through the hot gases will depend inter alia on the induced gas stream carrying them along.

Previous investigations (*1* to *3*) have been concerned with the entrainment of air into sprays produced from hollow-cone and solid-cone swirl spray nozzles. However, the complex nature of these systems has presented a series of experimental and theoretical difficulties and there are still a number of features about the processes of momentum transfer within a moving cloud of drops which remain to be elucidated.

In order to obtain a clearer picture of the detailed mechanism of air entrainment we have studied a flat spray, since this simplifies both the experimental and theoretical approach. Fan spray nozzles were utilized for this research because of the considerable information available on their characteristics (*4* to *11*).

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EXPERIMENTAL PROCEDURE

The factors which may be expected to affect the entrainment of air by a liquid spray are: the density and vapor pressure of the liquid; the spray velocity; drop size and size distribution; the liquid volume flow rate; the density and viscosity of the ambient gas stream; and ambient gas velocity. In the present research most of the experiments were carried out with a single liquid, while air at room temperature only was used as the ambient gas.

Apparatus

The experiments were carried out in a vertical 1-ft. × 1-ft., 6-in. rectangular air duct as shown in Figure 1. The entrance section was designed (*12*) to insure fully developed flow at the end of the contraction, while flow straighteners, 2½ in. long × ¼ in. wide, were inserted at the beginning of the parallel section. Three of the walls of the duct were fitted with twin-ground plate glass windows, while the interior of the fourth had a black nonreflecting surface.

The nozzle was mounted at the top of a vertical feed pipe to spray vertically upward, the liquid being introduced into the pipe by means of a tube placed across the duct below the flow straighteners. Test liquid was contained in a pressure vessel and ejected through the nozzle by means of compressed nitrogen, the nozzle being positioned such that the plane of the sheet was parallel to the two opposite windows. Air was drawn up through the duct past the nozzle by a centrifugal fan.

Cameras and associated optical and lighting equipment were mounted on an adjustable platform which could be placed at any height above the spray nozzle.